

Extremal Values of the Chromatic Number for a Given Degree Sequence

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Abstract

For a degree sequence $d : d_1 \geq \dots \geq d_n$, we consider the smallest chromatic number $\chi_{\min}(d)$ and the largest chromatic number $\chi_{\max}(d)$ among all graphs with degree sequence d . We show that if $d_n \geq 1$, then $\chi_{\min}(d) \leq \max \left\{ 3, d_1 - \frac{n+1}{4d_1} + 4 \right\}$, and, if $\sqrt{n + \frac{1}{4}} - \frac{1}{2} > d_1 \geq d_n \geq 1$, then $\chi_{\max}(d) = \max_{i \in [n]} \min \{i, d_i + 1\}$. For a given degree sequence d with bounded entries, we show that $\chi_{\min}(d)$, $\chi_{\max}(d)$, and also the smallest independence number $\alpha_{\min}(d)$ among all graphs with degree sequence d , can be determined in polynomial time.

Keywords: Degree sequence; chromatic number; independence number

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1 Introduction

We consider finite, simple, and undirected graphs. The *degree sequence* of a graph G with vertex set $\{v_1, \dots, v_n\}$ is the sequence $d_G(v_1), \dots, d_G(v_n)$ of its vertex degrees. A sequence d_1, \dots, d_n of integers is a *degree sequence* if it is the degree sequence of some graph. Repetitions within the degree sequence can be indicated by suitable exponents; the degree sequence of the star $K_{1,r}$ of order $r+1$, for instance, is $r, 1^r$. For a given sequence d , let $\mathcal{G}(d)$ be the set of all graphs G whose degree sequence is d ; called the *realizations* of d . For an integer n , let $[n]$ be the set of the positive integers at most n .

In the present paper we consider

$$\chi_{\min}(d) = \min \{\chi(G) : G \in \mathcal{G}(d)\} \quad \text{and} \quad \chi_{\max}(d) = \max \{\chi(G) : G \in \mathcal{G}(d)\}.$$

Punnim [11] determined $\chi_{\min}(d)$ and $\chi_{\max}(d)$ for regular degree sequences $d = r^n$ in almost all cases. The parameter $\chi_{\max}(d)$ was also considered by Dvořák and Mohar [3], who established degree sequence versions of the Hadwiger Conjecture and even the Hajós Conjecture, see also [14].

We contribute some bounds, exact values, and algorithmic results. Further discussion of related research will be given throughout the rest of the paper.

2 Some bounds and exact values

For a sequence d of non-negative integers $d_1 \geq \dots \geq d_n$, let $H(d)$ be the sequence

$$d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n.$$

Havel [9] and Hakimi [6] showed that d is a degree sequence if and only if $H(d)$ is a degree sequence. In fact, they observed that if d is a degree sequence, then there is a realization G of d in which the neighbours of a vertex of degree d_1 have degrees d_2, \dots, d_{d_1+1} . Iteratively applying this observation to a given degree sequence yields a realization that tends to contain a large complete subgraph on the vertices of large degrees, that is, such a realization may be expected to have high chromatic number.

In order to obtain a realization with hopefully small chromatic number, one can apply Havel and Hakimi's observation to the complement. More precisely, for a degree sequence d as above, the sequence \bar{d} defined as

$$n - 1 - d_n \geq \dots \geq n - 1 - d_1$$

is also a degree sequence; in fact, the graphs in $\mathcal{G}(\bar{d})$ are exactly the complements \bar{G} of the graphs G in $\mathcal{G}(d)$. Furthermore, by the above observation of Havel and Hakimi, \bar{d} has a realization in which the neighbors of a vertex of the largest degree $n - 1 - d_n$ have degrees $n - 1 - d_{n-1}, \dots, n - 1 - d_{d_n+1}$. Equivalently, as already observed by Kleitman and Wang [10] in a more general form, d has a realization in which the neighbors of a vertex of the smallest

degree d_n have degrees d_1, \dots, d_{d_n} . In summary, we obtain that d is a degree sequence if and only if the sequence $\bar{H}(d)$ defined as

$$d_1 - 1, \dots, d_{d_n} - 1, d_{d_n+1}, \dots, d_{n-1} \quad (1)$$

is a degree sequence. Iteratively applying this observation to a given degree sequence yields a realization that tends to avoid dense subgraphs on the vertices of large degrees, that is, such a realization may be expected to have small chromatic number.

As an example consider the degree sequence $d : r^{r+1}, 1^{r(r+1)}$ for some positive integer r . Havel and Hakimi's original observation yields the realization $K_{r+1} \cup \binom{r}{2} K_2$, whose chromatic number is $r + 1$, which equals $\chi_{\max}(d)$, while the above complementary version yields the realization $(r + 1)K_{1,r}$, whose chromatic number is 2, which equals $\chi_{\min}(d)$.

For a sequence d of integers d_1, \dots, d_n , let n be the *length* of d , let $\min(d) = \min\{d_1, \dots, d_n\}$, and let $\max(d) = \max\{d_1, \dots, d_n\}$. Furthermore, let $\bar{H}^0(d) = d$, $\bar{H}^1(d) = \bar{H}(d)$, and $\bar{H}^i(d) = \bar{H}(\bar{H}^{i-1}(d))$ for an integer i at least 2. Note that iteratively applying the reductions $d \mapsto H(d)$ or $d \mapsto \bar{H}(d)$ always requires reordering the constructed sequences in a non-increasing way.

Theorem 1 *If d is a degree sequence of length n , then*

$$\chi_{\min}(d) \leq \max \left\{ \min \left(\bar{H}^{n-i}(d) \right) : i \in [n] \right\} + 1.$$

Proof: Iteratively applying the complementary version of Havel and Hakimi's observation to the degree sequence d yields a realization G of d with vertex set $\{v_1, \dots, v_n\}$ such that, for i from n down to 1, the vertex v_i has degree $\min(\bar{H}^{n-i}(d))$ in the graph $G[\{v_1, \dots, v_i\}]$. Greedily coloring the vertices of G in the order v_1, \dots, v_n yields a coloring that uses at most $\max \left\{ \min \left(\bar{H}^{n-i}(d) \right) : i \in [n] \right\} + 1$ colors. \square

Note that for the degree sequence $d : r^{r+1}, 1^{r(r+1)}$ of length $n = (r + 1)^2$ considered as an example above, we obtain $\max \left\{ \min \left(\bar{H}^{n-i} \left(r^{r+1}, 1^{r(r+1)} \right) \right) : i \in [n] \right\} + 1 = 2$, that is, for this degree sequence d , Theorem 1 reproduces the correct value of $\chi_{\min}(d)$.

Unfortunately, Theorem 1 is not very explicit. As a more explicit consequence, we quantify how small degrees may reduce the effect of large degrees on $\chi_{\min}(d)$.

Corollary 2 *If d is a degree sequence $d_1 \geq \dots \geq d_n$, and k and ℓ are positive integers such that $d_k \geq k + \ell$ and $d_{n-\ell+1} \leq k$, then*

$$\chi_{\min}(d) \leq \max \left\{ d_1 - \frac{1}{k} \left(1 + \sum_{i=n-\ell+1}^n d_i \right) + 1, d_{k+1}, k \right\} + 1.$$

Proof: We consider the first ℓ applications of the reduction $d \mapsto \bar{H}(d)$. Since $d_k \geq k + \ell$ and $d_{n-\ell+1} \leq k$, we obtain that, for $i \in [\ell]$, the degree sequence $\bar{H}^i(d)$ arises from $\bar{H}^{i-1}(d)$ by removing the degree d_{n-i+1} , and reducing the d_{n-i+1} largest degrees by 1. For $i \in \{0, \dots, \ell\}$, let $\Delta_i = \max(\bar{H}^i(d))$, and let n_i be the number of entries of $\bar{H}^i(d)$ that are equal to Δ_i . Suppose,

for a contradiction, that $\Delta_\ell > \max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}\right\}$, where $D = \sum_{i=n-\ell+1}^n d_i$. Note that each of the $\ell + 1$ degree sequences $d, \bar{H}(d), \dots, \bar{H}^\ell(d)$ contains at most k entries that are strictly larger than d_{k+1} . So, for $i \in [\ell]$, we have

- $(\Delta_i, n_i) = (\Delta_{i-1}, n_{i-1} - d_{n-i+1})$ if $d_{n-i+1} < n_{i-1}$, and
- $\Delta_i = \Delta_{i-1} - 1$ and $n_i \leq k - (d_{n-i+1} - n_{i-1}) = n_{i-1} - d_{n-i+1} + k$ if $d_{n-i+1} \geq n_{i-1}$.

Note that $(k\Delta_{i-1} + n_{i-1}) - (k\Delta_i + n_i) \geq d_{n-i+1}$ in both cases. Summation over $i \in [\ell]$ yields $(k\Delta_0 + n_0) - (k\Delta_\ell + n_\ell) \geq D$. Since $\Delta_0 = d_1$, $n_0 \leq k$, and $n_\ell \geq 1$, this implies $\Delta_\ell \leq d_1 - \frac{D+1}{k} + 1$, which is a contradiction. Hence, $\Delta_\ell \leq \max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}\right\}$, and any realization H of the degree sequence $\bar{H}^\ell(d)$ can be colored using at most $\max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}\right\} + 1$ many colors. Adding ℓ further vertices of degrees $d_{n-\ell+1}, \dots, d_n$ one by one to H , and connecting them to suitable vertices according to the previous reductions, yields a realization G of d . Since the added vertices all have degree at most k , the coloring of H can be extended greedily to a coloring of G using at most $\max\left\{d_1 - \frac{D+1}{k} + 1, d_{k+1}, k\right\} + 1$ different colors in total. \square

For a given degree sequence d not satisfying any further restriction, one can only bound $\chi_{\min}(d)$ from above by $\max(d) + 1$. In fact, d might be $\max(d)^{\max(d)+1}, 0^{n-\max(d)-1}$, whose only realization contains a clique of size $\max(d) + 1$.

Our next two results improve this trivial estimate for graphs without isolated vertices.

Theorem 3 *If d is a degree sequence of length n with $\max(d) \geq \sqrt{\frac{n\delta}{4}}$ and $\min(d) \geq \delta$ for some positive integer δ , then $\chi_{\min}(d) \leq \max(d) - \frac{n\delta}{4\max(d)} + \delta + 3$.*

Proof: Our first goal is to show that we may assume that d has a realization with a very large independent set. Therefore, among all realizations G of the degree sequence d and all (not necessarily optimal) colorings f of G , we choose G and f with color classes V_1, \dots, V_k , where V_i contains n_i vertices for $i \in [k]$, in such a way that

- (n_1, \dots, n_k) is lexicographically maximal, and
- subject to this first condition, the number of edges between V_{k-1} and V_k is minimum.

Note that k may actually be larger than $\chi(G)$, and that n_1 is necessarily equal to the independence number $\alpha(G)$ of G .

Let $\Delta = \max(d)$. If $k \leq \Delta - \frac{n\delta}{4\Delta} + \delta + 3$, then $\chi_{\min}(d) \leq \chi(G) \leq k$ implies the desired bound. Hence, we may assume that $k > \Delta - \frac{n\delta}{4\Delta} + \delta + 3$. Since $\Delta \geq \sqrt{\frac{n\delta}{4}}$ and $\delta \geq 1$, we have $k \geq 5$. By the choice of the coloring f , there is an edge, say uv , between the smallest two color classes V_{k-1} and V_k . If $G \setminus (V_{k-1} \cup V_k \cup N_G(u) \cup N_G(v))$ contains an edge xy , then removing from G the two edges uv and xy , and adding the two edges ux and vy , yields another realization G' of d . Note that f is still a coloring of G' . This implies that there is a coloring f' of G' such that either the non-increasing vector of the sizes of the color classes is lexicographically larger than the one of f , or there are fewer edges between the two smallest color classes. Since both cases imply a

contradiction to the choice of G and f , we obtain that $V(G) \setminus (V_{k-1} \cup V_k \cup N_G(u) \cup N_G(v))$ is an independent set, which implies $\alpha(G) \geq n - (n_{k-1} + n_k) - 2\Delta$. Since V_{k-1} and V_k are the smallest two color classes, and $n_2 + \dots + n_k = n - \alpha(G)$, we obtain $n_{k-1} + n_k \leq \frac{2}{k-1}(n - \alpha(G))$. This implies $\alpha(G) \geq n - \frac{2}{k-1}(n - \alpha(G)) - 2\Delta$, and, using $k \geq 5$, we obtain $\alpha(G) \geq n - \frac{k-1}{k-3} \cdot 2\Delta \geq n - 4\Delta$.

Altogether, we may assume that d has a realization G with an independent set $I = \{u_1, \dots, u_\alpha\}$ of order at least $n - 4\Delta$. By the above-mentioned observations of Havel [9], Hakimi [6], Rao [12], and Kleitman and Wang [10], we may further assume that, for every $i \in [\alpha]$, the vertex u_i is adjacent to $d_G(u_i)$ vertices in $V(G) \setminus I$ of the largest degrees in the induced subgraph $G - \{u_1, \dots, u_{i-1}\}$ of G . Arguing as in the proof of Corollary 2, we obtain $((n - \alpha)\Delta + (n - \alpha)) - ((n - \alpha)\Delta(G - I) + 1) \geq d_G(u_1) + \dots + d_G(u_\alpha) \geq \alpha\delta$, where $\Delta(G - I)$ denotes the maximum degree of $G - I$. This implies $\Delta(G - I) \leq \Delta - \frac{\alpha\delta+1}{n-\alpha} + 1 \leq \Delta - \frac{(n-4\Delta)\delta+1}{4\Delta} + 1 = \Delta - \frac{n\delta+1}{4\Delta} + \delta + 1$. Therefore, we can color G using at most $\Delta - \frac{n\delta+1}{4\Delta} + \delta + 2$ colors on the vertices in $V(G) \setminus I$, and one additional color on the vertices in I , which implies $\chi_{\min}(d) \leq \chi(G) \leq \Delta - \frac{n\delta+1}{4\Delta} + \delta + 3$. \square

For positive integers r , s , and δ such that $r + 1$ is a multiple of δ , let d be the degree sequence $(r + s)^{r+1}, \delta^{s(r+1)/\delta}$. Since the sum of the largest $r + 1$ degrees equals exactly $2\binom{r+1}{2} + \delta s(r+1)/\delta$, every realization G of d contains a clique on the $r + 1$ vertices of largest degrees, and an independent set on the remaining vertices. Note that $\chi(G) \in \{r+1, r+2\}$, which, for $r \gg s \gg \delta$, is roughly $\max(d) - \frac{n \min(d)}{\max(d)}$, that is, up to the constants, the bound in Theorem 3 is best possible. In fact, by imposing a stronger lower bound on $\max(d)$ or by increasing the additive constant, the factor 4 within the term $\frac{n\delta+1}{4\Delta}$ can easily be reduced to slightly more than 2.

Our next result gives a best possible bound on $\chi_{\min}(d)$ for degree sequences of small degrees.

Theorem 4 *If n, d_1, \dots, d_n are integers such that $\sqrt{\frac{n-1}{2}} \geq d_1 \geq \dots \geq d_n \geq 1$ and $d_1 + \dots + d_n$ is even, then $\chi_{\min}(d) \leq 3$. (In particular, d_1, \dots, d_n is a degree sequence.)*

Proof: There is a partition of $[n]$ into two sets X and Y with $||X| - |Y|| \leq 1$ and $0 \leq s \leq d_1 \leq \sqrt{\frac{n-1}{2}}$, where $s = \sum_{i \in X} d_i - \sum_{i \in Y} d_i$; in fact, as long as there are two equal entries d_i and d_j in the sequence d_1, \dots, d_n , we assign i to X and j to Y , and remove d_i and d_j from the sequence, and once all remaining entries are distinct, say $d_{i_1} > \dots > d_{i_k}$, we assign i_1, i_3, \dots to X and i_2, i_4, \dots to Y . Let $x = |X|$ and $y = |Y|$. Note that $x, y \geq \frac{n-1}{2}$; in particular, $s \leq x$. Reducing s distinct entries of the sequence $(d_i)_{i \in X}$ by 1, and reordering yields a sequence $a_1 \geq \dots \geq a_x$. Reordering the sequence $(d_i)_{i \in Y}$ yields $b_1 \geq \dots \geq b_y$.

By construction, $\sum_{i \in [x]} a_i = \sum_{i \in [y]} b_i$, $\max\{a_1, b_1\} \leq \sqrt{\frac{n-1}{2}}$, and $b_y \geq 1$.

Let $k \in [x]$. If $k \leq \sqrt{\frac{n-1}{2}}$, then $a_1 \leq \sqrt{\frac{n-1}{2}}$ and $b_n \geq 1$ imply

$$\sum_{i \in [k]} a_i \leq ka_1 \leq \frac{n-1}{2} \leq y \leq \sum_{i \in [y]} \min\{k, b_i\}.$$

If $k > \sqrt{\frac{n-1}{2}}$, then $b_1 \leq \sqrt{\frac{n-1}{2}}$ implies

$$\sum_{i \in [k]} a_i \leq \sum_{i \in [x]} a_i = \sum_{i \in [y]} b_i = \sum_{i \in [y]} \min\{k, b_i\}.$$

By the Gale-Ryser Theorem [5, 15], there is a bipartite graph H with partite sets X and Y with $|X| = x$ and $|Y| = y$ such that the vertices in X have degrees a_1, \dots, a_x and the vertices in Y have degrees b_1, \dots, b_y . Since s has the same parity as $\sum_{i \in X} d_i + \sum_{i \in Y} d_i = d_1 + \dots + d_n$, it is an even integer, and adding to H a matching of size $s/2$ incident to those vertices in X corresponding to the entries of $(d_i)_{i \in X}$ that were previously reduced by 1, results in a graph G with degree sequence d_1, \dots, d_n . Clearly, $\chi(G) \leq 3$, and the upper bound on $\chi_{\min}(d)$ follows. \square

The conclusion of Theorem 4 is best possible, because there might not be a subset X of $[n]$ with $\sum_{i \in X} d_i = \sum_{i \in [n] \setminus X} d_i$, which is a necessary condition for the existence of a bipartite realization. The complexity of deciding the existence of a bipartite realization for a given degree sequence is unknown.

Note that together, Theorem 3 and Theorem 4 imply

$$\chi_{\min}(d) \leq \max \left\{ 3, \max(d) - \frac{n+1}{4 \max(d)} + 4 \right\}$$

for every degree sequence d with $\min(d) \geq 1$.

Theorem 4 has the following variant where the essential assumption is that $\max(d) - \min(d)$ is small. Note that this next result also covers regular degree sequences of sufficient length.

Theorem 5 *If n, d_1, \dots, d_n are integers and $\epsilon > 0$ is such that $\frac{n-1}{2}\epsilon \geq d_1 \geq \dots \geq d_n \geq 1$, $d_1 - d_n \leq \sqrt{\frac{n-1}{2}}(1 - \epsilon)$, and $d_1 + \dots + d_n$ is even, then $\chi_{\min}(d) \leq 3$.*

Proof: We may assume that $d_1 > \sqrt{\frac{n-1}{2}}$; otherwise Theorem 4 implies the result. Furthermore, we have $\epsilon \leq 1$. Exactly as in the proof of Theorem 4, we obtain the existence of a partition of $[n]$ into two sets X and Y with $||X| - |Y|| \leq 1$ and $0 \leq s \leq d_1 \leq \frac{n-1}{2}\epsilon$, where $s = \sum_{i \in X} d_i - \sum_{i \in Y} d_i$. Setting $x = |X|$ and $y = |Y|$, we obtain, as above, that $x, y \geq \frac{n-1}{2}$, $s \leq x$, and s is even. Let $a_1 \geq \dots \geq a_x$ and $b_1 \geq \dots \geq b_y$ be as in the proof of Theorem 4. By construction, $\sum_{i \in [x]} a_i = \sum_{i \in [y]} b_i$, $\max\{a_1, b_1\} \leq d_1$, and $b_y \geq d_n$.

Notice that as $d_1 > \sqrt{\frac{n-1}{2}}$, we have

$$\frac{d_n}{d_1} \geq \frac{d_1 - \sqrt{\frac{n-1}{2}}(1 - \epsilon)}{d_1} \geq 1 - (1 - \epsilon) = \epsilon.$$

Let $k \in [x]$. If $k \leq d_n$, then

$$\sum_{i \in [k]} a_i \leq kd_1 \leq k \frac{n-1}{2} \leq ky \leq \sum_{i \in [y]} \min\{k, b_i\}.$$

If $d_n < k < d_1$, then

$$\sum_{i \in [k]} a_i \leq kd_1 \leq d_1^2 \leq \frac{n-1}{2} \epsilon d_1 \leq \frac{n-1}{2} d_n \leq yd_n \leq \sum_{i \in [y]} \min\{k, b_i\}.$$

And, if $k \geq d_1$, then

$$\sum_{i \in [k]} a_i \leq \sum_{i \in [x]} a_i = \sum_{i \in [y]} b_i = \sum_{i \in [y]} \min\{k, b_i\}.$$

At this point, the proof can be completed exactly as the proof of Theorem 4. \square

For a graph G with degree sequence $d_1 \geq \dots \geq d_n$, Welsh and Powell [16] observed

$$\chi(G) \leq \max_{i \in [n]} \min\{i, d_i + 1\}, \quad (2)$$

which is an immediate consequence of applying the natural greedy coloring algorithm to the vertices of G in an order of non-increasing degrees. If $d_1 \geq \dots \geq d_n$ is a degree sequence such that $d_p - d_{p+1} \geq p - 2$ for $p = \max_{i \in [n]} \min\{i, d_i + 1\}$, then Havel and Hakimi's observation explained above implies the existence of a realization G of d for which the vertices of degrees d_1, \dots, d_p form a clique. This implies $p \leq \chi(G) \leq \chi_{\max}(d) \leq p$, that is, $\chi_{\max}(d) = \max_{i \in [n]} \min\{i, d_i + 1\}$ for such degree sequences.

Our next result shows that the Welsh-Powell bound (2) also gives the correct value of $\chi_{\max}(d)$ for degree sequences d of small degrees.

Theorem 6 *If n, d_1, \dots, d_n are integers such that $\sqrt{n + \frac{1}{4}} - \frac{1}{2} > d_1 \geq \dots \geq d_n \geq 1$ and $d_1 + \dots + d_n$ is even, then $\chi_{\max}(d) = \max_{i \in [n]} \min\{i, d_i + 1\}$.*

Proof: Let $p = \max_{i \in [n]} \min\{i, d_i + 1\}$. Note that $p \leq d_p + 1 \leq d_1 + 1$.

By the Welsh-Powell bound (2), every graph G with degree sequence d_1, \dots, d_n satisfies $\chi(G) \leq p$, which implies $\chi_{\max}(d) \leq p$. In order to establish equality, we show the existence of a realization that contains a clique of size p .

Let $k \in [n]$. We obtain $\sum_{i \in [k]} d_i \leq kd_1$ and $k(k-1) + \sum_{i \in [n] \setminus [k]} \min\{k, d_i\} \geq k(k-1) + n - k$. Therefore, $\sum_{i \in [k]} d_i$ is at most $k(k-1) + \sum_{i \in [n] \setminus [k]} \min\{k, d_i\}$ if $kd_1 \leq k(k-1) + n - k$, which is equivalent to $k(d_1 + 2 - k) \leq n$. Since $\sqrt{n + \frac{1}{4}} - \frac{1}{2} > d_1 \geq 1$ implies $n \geq 3$ and $k(d_1 + 2 - k) \leq \left(\frac{d_1 + 2}{2}\right)^2 \leq n$, the Erdős-Gallai Theorem [4] implies the existence of a graph with degree sequence d_1, \dots, d_n . Among all such graphs with vertex set $\{v_1, \dots, v_n\}$, where v_i has degree d_i for $i \in [n]$, we choose G such that the number $m(G[\{v_1, \dots, v_p\}])$ of edges of the subgraph of G induced by $\{v_1, \dots, v_p\}$ is as large as possible.

Suppose, for a contradiction, that $G[\{v_1, \dots, v_p\}]$ is not a clique, that is, v_i and v_j are not adjacent in G for distinct i and j in $[p]$. By the choice of p , we have $d_i, d_j \geq p - 1$, which implies that v_i and v_j both have at least one neighbor in $R = \{v_{p+1}, \dots, v_n\}$.

First, we assume that v_i and v_j both have the same unique neighbor v_r in R , that is, $\{v_r\} = N_G(v_i) \cap R = N_G(v_j) \cap R$. Since there are at most $1 + d_1^2$ vertices at distance at most 2

from v_r , including, in particular, v_i and v_j , and $n - (p - 2) - (1 + d_1^2) \geq n - d_1^2 - d_1 > 0$, there is a vertex v_s in R with a neighbor v_t such that v_s and v_t are both not adjacent to v_r . Now, removing from G the edges $v_i v_r$, $v_j v_r$, and $v_s v_t$, and adding the edges $v_i v_j$, $v_r v_s$, and $v_r v_t$ yields a realization G' of d_1, \dots, d_n with $m(G'[\{v_1, \dots, v_p\}]) > m(G[\{v_1, \dots, v_p\}])$, which contradicts the choice of G .

Now, we may assume that v_i is adjacent to some vertex v_r in R , and that v_j is adjacent to a different vertex v_s in R . If v_r is not adjacent to v_s , then removing from G the edges $v_i v_r$ and $v_j v_s$, and adding the edges $v_i v_j$ and $v_r v_s$ yields a realization G' of d_1, \dots, d_n with $m(G'[\{v_1, \dots, v_p\}]) > m(G[\{v_1, \dots, v_p\}])$, which contradicts the choice of G . Hence, we may assume that v_r and v_s are adjacent. Since there are at most $1 + d_1^2$ vertices at distance at most 2 from v_r , including, in particular, v_i , v_s , and v_j , and $n - (p - 2) - (1 + d_1^2) \geq n - d_1^2 - d_1 > 0$, there is a vertex v_p in R with a neighbor v_q such that v_p is not adjacent to v_s , and v_q is not adjacent to v_r . Note that v_q may be v_j , in which case, v_j has distance 2 from v_r . Now, removing from G the edges $v_i v_r$, $v_j v_s$, and $v_p v_q$, and adding the edges $v_i v_j$, $v_s v_p$, and $v_r v_q$ yields a realization G' of d_1, \dots, d_n with $m(G'[\{v_1, \dots, v_p\}]) > m(G[\{v_1, \dots, v_p\}])$, which contradicts the choice of G .

Altogether, we obtain that G contains a clique of order p , which completes the proof. \square

3 Algorithmic aspects

One way to establish that $\chi_{\max}(d)$ is large is to show the existence of a realization of d that contains a large clique. Dvořák and Mohar [3] proved the best possible statement that for every degree sequence d , some realization of d has a clique of size at least $5/6(\chi_{\max}(d) - 3/5)$. Since Rao [12, 13] efficiently characterized the largest clique size $\omega_{\max}(d)$ of any realization of a given degree sequence d , and, trivially, $\chi_{\max}(d) \geq \omega_{\max}(d)$, we immediately obtain that $\chi_{\max}(d)$ can be approximated in polynomial time for a given d within an asymptotic factor of $6/5$.

Our next two results show that $\chi_{\max}(d)$ and $\chi_{\min}(d)$ can both be determined in polynomial time for given degree sequences with bounded entries.

Corollary 7 *Let Δ be a fixed positive integer.*

For a given degree sequence d with $\max(d) \leq \Delta$, one can determine $\chi_{\max}(d)$ in polynomial time.

Proof: Let d have length n . Clearly, we may assume $\min(d) \geq 1$. If $\sqrt{n-2} \geq \Delta$, then Theorem 6 implies that $\chi_{\max}(d)$ coincides with the Welsh-Powell bound (2). If $\sqrt{n-2} < \Delta$, then, as Δ is fixed, there are only constantly many realizations of d , which can all be generated and optimally colored by brute force in constant time. \square

Theorem 8 *Let k and p be fixed positive integers.*

For a given degree sequence d with at most p distinct entries, one can decide in polynomial time whether $\chi_{\min}(d) \leq k$.

Proof: Let $d : d_1^{n_1}, \dots, d_p^{n_p}$ and $n = n_1 + \dots + n_p$. There are $\prod_{i=1}^p \binom{n_i+k-1}{k-1} \leq \left(\frac{n}{p} + k\right)^{kp}$ distinct matrices $(n_i^j)_{(i,j) \in [p] \times [k]}$ with non-negative integral entries n_i^j such that $\sum_{j=1}^k n_i^j = n_i$ for $i \in [p]$. It is easy to see that $\chi_{\min}(d) \leq k$ if and only if there is such a matrix $(n_i^j)_{(i,j) \in [p] \times [k]}$ for which the complete k -partite graph whose j th partite set V_j has order $\sum_{i=1}^p n_i^j$ for $j \in [k]$, has a factor G such that V_j contains exactly n_i^j vertices of degree d_i in G for every $i \in [p]$ and $j \in [k]$. Since the existence of such a factor can be decided in polynomial time using matching methods, and, for fixed k and p , there are only polynomially many different suitable matrices, the desired statement follows. \square

It seems plausible to wonder whether $\chi_{\max}(d)$ is linked to $\alpha_{\min}(d)$, the minimum independence number of a realization of d . While $\alpha_{\max}(d) = \omega_{\max}(\vec{d})$ can be determined efficiently using the results of Rao [12, 13], Bauer, Hakimi, Kahl, and Schmeichel [1] conjectured that it is computationally hard to determine $\alpha_{\min}(d)$ for a given degree sequence d .

Our next goal is to show that also $\alpha_{\min}(d)$ can be determined in polynomial time for given degree sequences d with bounded entries.

For a degree sequence d_1, \dots, d_n , let $\alpha_{CW}(d) = \sum_{i=1}^n \frac{1}{d_i+1}$. Caro [2] and Wei [17] proved that $\alpha(G) \geq \alpha_{CW}(d)$ for every graph G with degree sequence d . For a connected graph G with degree sequence d , Harant and Rautenbach [7] showed $\alpha(G) \geq k \geq \sum_{u \in V(G)} \frac{1}{d_G(u) - f(u) + 1}$, where k is an integer, and, for every vertex u of G , $f(u)$ is a non-negative integer at most $d_G(u)$ such that $\sum_{u \in V(G)} f(u) \geq 2(k-1)$. This improved an earlier result of Harant and Schiermeyer [8].

If $\alpha_{CW}(d) \geq 2$, then $k \geq \alpha_{CW}(d)$ implies $2(k-1) \geq k \geq \alpha_{CW}(d)$, and, hence,

$$\begin{aligned} \alpha(G) &\geq \sum_{u \in V(G)} \frac{1}{d_G(u) - f(u) + 1} \\ &= \alpha_{CW}(d) + \sum_{u \in V(G)} \left(\frac{1}{d_G(u) - f(u) + 1} - \frac{1}{d_G(u) + 1} \right) \\ &\geq \alpha_{CW}(d) + \frac{1}{(\max(d) + 1)^2} \sum_{u \in V(G)} f(u) \\ &\geq \left(1 + \frac{1}{(\max(d) + 1)^2} \right) \alpha_{CW}(d). \end{aligned}$$

Theorem 9 *Let Δ be a fixed positive integer.*

For a given degree sequence d with $\max(d) \leq \Delta$, every component of every realization G of d with $\alpha(G) = \alpha_{\min}(d)$ has order at most $((\Delta + 1)^3 + 1) \left(\left(\frac{\Delta+2}{2} \right)^2 + \binom{\Delta+1}{2} \right)$. In particular, one can determine $\alpha_{\min}(d)$ in polynomial time.

Proof: Let d be a degree sequence with $\max(d) \leq \Delta$. Let G be a realization of d with $\alpha(G) = \alpha_{\min}(d)$. Suppose, for a contradiction, that some component K of G has order $n(K)$ more than the stated value. Let R be a set of $\left(\frac{\Delta+2}{2} \right)^2$ vertices of K . For $i \in [\Delta]$, let V_i be the set of vertices of degree i in $V(K) \setminus R$, and let $n_i = |V_i|$. Let $p_i = \left\lfloor \frac{n_i}{i+1} \right\rfloor$, and let S_i arise by removing

$p_i(i+1)$ vertices from V_i for each $i \in [\Delta]$. Note that $|S| \leq \sum_{i=1}^{\Delta} i = \binom{\Delta+1}{2}$, where $S = S_1 \cup \dots \cup S_{\Delta}$, that is, $R \cup S$ is a set of at least $\left(\frac{\Delta+2}{2}\right)^2$ and at most $\left(\frac{\Delta+2}{2}\right)^2 + \binom{\Delta+1}{2}$ many vertices of K . Let d' be the sequence of the degrees of the vertices in $R \cup S$, and let d'' be the sequence of the degrees of the vertices in $V(K) \setminus (R \cup S)$. Note that $\alpha_{CW}(d'') \geq \frac{(n(K) - |R \cup S|)}{\Delta+1}$. Hence, the lower bound on $n(K)$ implies $\left(1 + \frac{1}{(\Delta+1)^2}\right) \alpha_{CW}(d'') = \frac{1}{(\Delta+1)^2} \alpha_{CW}(d'') + \alpha_{CW}(d'') > |R \cup S| + \alpha_{CW}(d'')$. As observed in the proof of Theorem 6, the Erdős-Gallai Theorem implies that the sequence d' , which is a sequence of positive integers at most Δ that is of length at least $\left(\frac{\Delta+2}{2}\right)^2$, is a degree sequence. Let K'_0 be a realization of d' . By construction, the graph $K' = K'_0 \cup \bigcup_{i=1}^{\Delta} p_i K_{i+1}$ has exactly the same degree sequence as K . By the result of Harant and Rautenbach mentioned above,

$$\begin{aligned}
\alpha(K') &= \alpha(K'_0) + \sum_{i=1}^{\Delta} p_i \alpha(K_{i+1}) \\
&= \alpha(K'_0) + \alpha_{CW}(d'') \\
&\leq |R \cup S| + \alpha_{CW}(d'') \\
&< \left(1 + \frac{1}{(\Delta+1)^2}\right) \alpha_{CW}(d'') \\
&< \left(1 + \frac{1}{(\Delta+1)^2}\right) \alpha_{CW}(d) \\
&\leq \alpha(K).
\end{aligned}$$

Therefore, replacing K by K' within G yields a realization G' of d with $\alpha(G') < \alpha(G)$, contradicting the choice of G . This completes the proof of the first part of the statement.

Since, as Δ is fixed, there are only finitely many graphs of maximum degree at most Δ and order at most $((\Delta+1)^3 + 1) \left(\left(\frac{\Delta+2}{2}\right)^2 + \binom{\Delta+1}{2}\right)$. Listing, for each of these graphs, the degree sequence and the independence number, it is a routine matter to determine $\alpha_{\min}(d)$ for a given degree sequence d with $\max(d) \leq \Delta$ by dynamic programming in polynomial time. \square

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